CONVERGENCE OF BROWNIAN MOTIONS ON $\mathrm{RCD}(K,\infty)$ SPACES

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ABSTRACT. Suppose that metric measure spaces $\mathcal{X}_n = (X_n, d_n, m_n)$ satisfy $\mathrm{RCD}(K,\infty)$ conditions with $m_n(X_n) = 1$. Then the measured Gromov convergence (introduced by Gigili–Mondino–Savaré [15]) of $\mathcal{X}_n$ is equivalent to the weak convergence of the laws of Brownian motions on $\mathcal{X}_n$ with initial distributions m_n.

1. INTRODUCTION

This paper is a continuation of our paper [26] which focused on the weak convergence of Brownian motions on metric measure spaces satisfying $\mathrm{RCD}^*(K,N)$ conditions. In [26], we showed that the measured Gromov–Hausdorff convergence of normalized metric measure spaces is equivalent to the weak convergence of the laws of Brownian motions under $\mathrm{RCD}^*(K,N)$ conditions with uniform diameter bounds. In this paper, we study the weak convergence of Brownian motions on $\mathrm{RCD}(K,\infty)$ spaces, which is more general framework than $\mathrm{RCD}^*(K,N)$ spaces where dimensional upper bounds are not necessarily assumed. We aim to answer the following question:

(Q) Does the weak convergence of Brownian motions follow from some convergence of the underlying spaces?

We assume the following conditions:

Assumption 1.1. Let $K \in \mathbb{R}$. For $n \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, let $\mathcal{X}_n = (X_n, d_n, m_n)$ be a sequence of metric measure spaces satisfying the $\mathrm{RCD}(K,\infty)$ condition with $m_n(X_n) = 1$ and $\mathrm{supp}[m_n] = X_n$.

The notion of $\mathrm{RCD}(K,\infty)$ spaces was introduced by Ambrosio–Gigli–Savaré [5] and Ambrosio–Gigli–Mondino–Rajala [3], and this is a generalization of Ricci $\geq K$ for non-smooth spaces with Cheeger energies being quadratic. We have such examples as measured Gromov-Hausdorff limit spaces of Riemannian manifolds with lower Ricci curvature bounds, Alexandrov spaces with lower curvature bounds (Petrunin [21] and Zhang–Zhu [28]), Hilbert spaces with log-concave measures (Ambrosio–Savaré–Zambotti [7]), and configuration spaces on Riemannian manifolds with lower Ricci curvature bounds (Erbar–Huesmann [11]). We briefly recall about the $\mathrm{RCD}(K,\infty)$ spaces in Section 2.5.

Under Assumption 1.1, it is known that there exists a conservative diffusion process on $\mathcal{X}_n$ associated with the Cheeger energy and unique in quasi-every starting points in X_n (see [5]). We denote it by $(\{\mathbb{P}_n^x\}_{x \in X_n}, \{B_t^n\}_{t \geq 0})$,

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called *Brownian motion on* $\mathcal{X}_n$. As the main result, we show that *the measured Gromov convergence* (introduced by Gigli–Mondino–Savaré [15]) of the underlying spaces $\mathcal{X}_n$ is equivalent to the weak convergence of the laws of Brownian motions on $\mathcal{X}_n$ with initial measures m_n. To be precise, we have the following result:

Theorem 1.2. *Suppose that Assumption 1.1 holds. Then the following (i) and (ii) are equivalent:*

(i) *$\mathcal{X}_n$ converges to $\mathcal{X}_\infty$ in the measured Gromov sense;*

(ii) *There exist*

$$\begin{cases} \textit{a complete separable metric space } (X,d) \\ \textit{isometric embeddings } \iota_n : X_n \to X \ (n \in \overline{\mathbb{N}}) \end{cases}$$

such that

$$\iota_n(B^n_\cdot)_\#\mathbb{P}^{m_n}_n \to \iota_\infty(B^\infty_\cdot)_\#\mathbb{P}^{m_\infty}_\infty \quad \textit{weakly} \quad \textit{in } \mathcal{P}(C([0,\infty);X)).$$

See the definition of the measured Gromov convergence in [15, §3], which will be stated in Section 2.3 in the present paper. Note that, in general, the measured Gromov convergence is weaker than the measured Gromov–Hausdorff convergence. Here we mean

$$\mathbb{P}^{m_n}_n(A) := \int_{X_n} \mathbb{P}^x_n(A) m_n(dx), \quad \forall A \subset X_n \quad \text{Borel}. \tag{1.1}$$

Here $\mathcal{P}(C([0,\infty);X))$ denotes the set of all probability measures on $\big(C([0,\infty);X),\delta\big)$, which means the set of continuous functions from $[0,\infty)$ to X with a distance δ inducing the uniform convergence on compact sets (see (2.1)). The subscript # means the operation of the push-forward of measures.

Remark 1.3. We give several remarks for Theorem 1.2.

(i) In Theorem 1.2, the implication $(ii) \implies (i)$ is trivial because the weak convergence of $\iota_n(B^n_\cdot)_\#\mathbb{P}^{m_n}_n$ to $\iota_\infty(B^\infty_\cdot)_\#\mathbb{P}^{m_\infty}_\infty$ clearly implies the weak convergence of the initial distributions $\iota_{n\#}m_n$ to $\iota_{\infty\#}m_\infty$. This implies the statement (i) by the definition of the measured Gromov convergence (see Section 2.3).

(ii) In contrast to our result [26, Theorem 1.2], Theorem 1.2 in the present paper restricts the initial measures with the underlying measure m_n. This is because we use the Lyons–Zheng decomposition to show the tightness of Brownian motions. See Theorem 2.9 and Subsection 1.1.

The present paper is organized as follows: In Section 2, we prepare notation and preliminary facts. In Section 3, we give a proof of Theorem 1.2. We finish the introduction by a sketch of proof of Theorem 1.2:

1.1. **Sketch of proof of Theorem 1.2.** The implication (ii) $\implies$ (i) is trivial by definition of the measured Gromov convergence. We only have to show the implication (i) $\implies$ (ii). It is enough to show *the tightness* (Lemma 3.1) and *the weak convergence of the laws of finite-dimensional distributions* (CFD) (Lemma 3.2).

Tightness (Lemma 3.1): In contrast to $\mathrm{RCD}^*(K,N)$ spaces, $\mathrm{RCD}(K,\infty)$ spaces do not have good heat kernel estimates (like the Gaussian estimate),

and there is no hope to show the tightness by using heat kernel estimates. Alternatively, we use *the Lyons-Zheng decomposition* for Dirichlet forms ([18]) for every bounded Lipschitz function h on the ambient space X. The key point is to estimate the quadratic variation of the martingale part $M_t^{[h]}$ with respect to Fukushima's decomposition of $h(B_t) - h(B_0)$. We can estimate $M_t^{[h]}$ uniformly in n by means of the following representation

$$M_t^{[h]} = \int_0^t |\nabla h|^2(B_u^n)du,$$

and $|\nabla h| \le \mathrm{Lip}_{X_n}(h) \le \mathrm{Lip}_X(h)$ since X_n's are *isometrically* embedded into X.

CFD (Lemma 3.2): This is proved by using the Mosco convergence of Cheeger energies of Gigli–Mondino–Savaré [15].

2. Notation & Preliminary Results

2.1. **Notation.** Let $\mathbb{N} = \{0, 1, 2, ...\}$ and $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$.

Let (X, d) be a complete separable metric space. We denote by $\mathscr{B}(X)$ the family of all Borel sets in (X, d). We write $B_r(x) = \{y \in X : d(x, y) < r\}$ for a open ball centered at $x \in X$ with radius $r > 0$. Let $C(X)$ denote the set of real-valued continuous functions on X. Let $C_b(X), C_0(X)$ and $C_{bs}(X)$ denote subspaces of $C(X)$ consisting of bounded functions, functions with compact support, and bounded functions with bounded support, respectively. We denote by $\mathcal{B}_b(X)$ the set of real-valued bounded Borel-measurable functions on X. Let $\mathcal{P}(X)$ denote the set of Borel probability measures on X. We denote by $C([0, \infty), X)$ the set of continuous functions on $[0, \infty)$ to X. Let δ denote the following distance for $v, w \in C([0, \infty), X)$

$$\delta(v, w) = \int_{[0,\infty)} e^{-T}(1 \wedge \sup_{t\in[0,T]} d(v(t), w(t)))dT. \tag{2.1}$$

It is known that δ induces the topology of the uniform convergence on compact sets, and a metric space $(C([0, \infty, X), \delta)$ becomes a complete separable metric space. We say that $\gamma : [a, b] \to X$ is a *curve on* X if $\gamma : [a, b] \to X$ is continuous. A curve $\gamma : [a, b] \to X$ is said to be *connecting* x *and* y if $\gamma_a = x$ and $\gamma_b = y$. A curve $\gamma : [a, b] \to X$ is said to be *a minimal geodesic* if

$$d(\gamma_t, \gamma_s) = \frac{|s - t|}{|b - a|} d(\gamma_a, \gamma_b) \quad a \le t \le s \le b.$$

In particular, if $\frac{d(\gamma_a,\gamma_b)}{|b-a|}$ can be replaced to 1, we say that γ is *unit-speed.*

In this paper, we say that (X, d, m) is *a metric measure space* if

- (X, d) is a complete separable metric space;
- m is a non-zero Borel measure on X which is locally finite in the sense that $m(B_r(x)) < \infty$ for all $x \in X$ and sufficiently small $r > 0$.

Let $\mathrm{supp}[m] = \{x \in X : m(B_r(x)) > 0,\ \forall r > 0\}$ denote the support of m. Let (Y, d_Y) be a complete separable metric space. For a Borel measurable map $f : X \to Y$, let $f_\# m$ denote the push-forward measure on Y:

$$f_\# m(B) = m(f^{-1}(B)) \quad \text{for any Borel set} \quad B \in \mathscr{B}(Y).$$

2.2. **L^2-Wasserstein Space.** Let (X_i, d_i) $(i = 1, 2)$ be complete separable metric spaces. For $\mu_i \in \mathcal{P}(X_i)$, a probability measure $q \in \mathcal{P}(X_1 \times X_2)$ is called *a coupling of* μ_1 *and* μ_2 if

$$\pi_{1\#}q = \mu_1 \quad \text{and} \quad \pi_{2\#}q = \mu_2,$$

where π_i $(i = 1, 2)$ is the projection $\pi_i : X_1 \times X_2 \to X_i$ as $(x_1, x_2) \mapsto x_i$. We denote by $\Pi(\mu, \nu) \subset \mathcal{P}(X \times X)$ the set of all couplings of μ and ν. For a complete separable metric space (X, d), let $\mathcal{P}_2(X)$ be the subset of $\mathcal{P}(X)$ consisting of all Borel probability measures μ on X with finite second moment:

$$\int_X d^2(x, \overline{x}) d\mu(x) < \infty \quad \text{for some (and thus any) } \overline{x} \in X.$$

We endow $\mathcal{P}_2(X)$ with the quadratic transportation distance W_2, called *L^2-Wasserstein distance*, defined as follows:

$$W_2(\mu, \nu) = \Big(\inf_{q \in \Pi(\mu,\nu)} \int_{X \times X} d^2(x, y) dq(x, y) \Big)^{1/2}. \tag{2.2}$$

A coupling $q \in \Pi(\mu, \nu)$ is called *an optimal coupling* if q attains the infimum in the equality (2.2). It is known that, for any μ, ν, there always exists an optimal coupling q of μ and ν (e.g., [27, §4]). It is known that $(\mathcal{P}_2(X), W_2)$ is a complete separable metric space (e.g., [27, Theorem 6.18]).

2.3. **Measured Gromov convergence.** In this subsection, following [15, §3], we recall the *measured Gromov convergence*. We sometimes write *mG-convergence* for short. In [15, §3], they defined the mG-convergence for metric measure spaces possibly allowing unbounded measures, but it is enough for this paper to recall the mG-convergence just for metric measure spaces with normalized measures.

Let (X, d, m) be a normalized metric measure space, that is,

- (X, d, m) is a metric measure space;
- $m(X) = 1$.

Two metric measure spaces (X_1, d_1, m_1) and (X_2, d_2, m_2) are said to be *isomorphic* if there exists an isometry $\iota : \mathrm{supp}[m_1] \to \mathrm{supp}[m_2]$ such that

$$\iota_\# m_1 = m_2.$$

Fix a continuous, concave and non-constant function $c : [0, \infty) \to \mathbb{R}$ so that

$$c(0) = 0, \quad \lim_{r \to \infty} c(r) < \infty.$$

Typical examples are $c(r) = \tanh(r)$ and $c(r) = r \wedge 1$. Let a cost function $c(x, y)$ be defined as follows:

$$c(x, y) := c(d(x, y)), \quad x, y \in X.$$

Then it is easy to check that $c(x, y)$ is a complete and bounded distance on X inducing the same topology as that of d. Now we introduce the Kantorovich–Rubinstein–Wasserstein distance W_c as follows: for $\mu, \nu \in \mathcal{P}(X)$,

$$W_c(\mu, \nu) := \inf_{q \in \Pi(\mu,\nu)} \int_X c(x, y) dq(x, y).$$

Regardless of choices of c, we know that $(\mathcal{P}(X), W_c)$ becomes a complete separable metric space which induces the same topology as the topology of the weak convergence of probability measures (see [27, Chapter 6]).

Now we recall the measured Gromov convergence.

Definition 2.1 (§3 in [15])**.** For two normalized metric measure spaces (X_1, d_1, m_1) and (X_2, d_2, m_2),

$$\mathbb{G}_W((X_1, d_1, m_1), (X_2, d_2, m_2)) := \inf W_c(\iota_{1\#}m_1, \iota_{2\#}m_2),$$

where the infimum is taken among all complete separable metric spaces (X, d) and all isometric embeddings $\iota_1 : X_1 \to X$ and $\iota_2 : X_2 \to X$. We say that a sequence $\mathcal{X}_n = (X_n, d_n, m_n)$ converges to $\mathcal{X}_\infty = (X_\infty, d_\infty, m_\infty)$ *in the measured Gromov sense* if

$$\mathbb{G}_W(\mathcal{X}_n, \mathcal{X}_\infty) \to 0 \quad \text{as} \quad n \to \infty.$$

Remark 2.2. In [15], they introduced a more general version of mG-convergence for general pointed metric measure spaces (*pointed measured Gromov convergence*) whose totall masses are not necessarily finite.

We know the following equivalence:

Proposition 2.3 (Theorem 3.15 in [15])**.** *Let $\{(X_n, d_n, m_n)\}_{n\in\overline{\mathbb{N}}}$ be a sequence of normalized metric measure spaces. Then the following are equivalent:*

(i) $(X_n, d_n, m_n) \stackrel{mG}{\to} (X_\infty, d_\infty, m_\infty)$ *as* $n \to \infty$;
(ii) *There exists a complete separable metric space (X, d) and isometric embeddings $\iota_n : X_n \to X$ for $n \in \overline{\mathbb{N}}$ so that*

$$\iota_{n\#}m_n \to \iota_{\infty\#}m_\infty \quad \textit{weakly in } \mathcal{P}(X).$$

2.4. **Cheeger's L^2-energy functional.** In this subsection, we follow [5] to define Cheeger's L^2-energy functionals on metric measure spaces.

Let (Z, d_Z) be a complete separable metric space and $I \subset \mathbb{R}$ be a non-trivial interval. A curve $I \ni t \mapsto z_t \in Z$ is absolutely continuous if there exists a non-negative function $f \in L^1(I, dt)$ such that (dt denotes the Lebesgue measure on I)

$$d_Z(z_t, z_s) \le \int_t^s f(r)dr \quad \forall t, s \in I,\ t < s. \tag{2.3}$$

For an absolutely continuous curve z_t, the limit $\lim_{h\to 0} \frac{d_Z(z_{t+h}, z_t)}{|h|}$ exists for a.e. $t \in I$ and, this limit defines an $L^1(I, dt)$ function. We denote $\lim_{h\to 0} \frac{d_Z(z_{t+h}, z_t)}{|h|}$ by $|\dot{z}_t|$ called *the metric speed at t*. Note that the metric speed $|\dot{z}_t|$ is the minimal function in the a.e. sense among $L^1(I, dt)$ functions satisfying (2.3) (see [6]). We denote by $\mathrm{AC}^p(I; Z)$ the set of all absolutely continuous curves with their metric derivatives in $L^p(I)$. Let $C(I; Z)$ denote the set of continuous functions from I to Z. Define a map

$$C(I; Z) \ni \gamma \mapsto \mathcal{E}_2[\gamma] = \begin{cases} \displaystyle\int_I |\dot{\gamma}_t|^2 dt, & \text{if} \quad \gamma \in \mathrm{AC}^2(I; Z), \\ +\infty & \text{otherwise.} \end{cases}$$

Given an absolutely continuous curve $\mu \in \mathrm{AC}(I; (\mathcal{P}_2(Z), W_2))$, we denote by $|\dot{\mu}_t|$ its metric speed in the space $(\mathcal{P}_2(Z), W_2)$. Let $e_t : C(I; Z) \to Z$ be the evaluation map $e_t(\gamma) = \gamma_t$. If $\pi \in \mathcal{P}(C(I; Z))$ satisfies $(e_t)_\# \pi = \mu_t$ for any $t \in I$, it is easy to see that

$$\int_I |\dot{\mu}_t| dt \le \int \mathcal{E}_2[\gamma] d\pi(\gamma).$$

Let (X, d, m) be a metric measure space. We now recall notions of *test plan, weak upper gradient, Sobolev class and Cheeger energy* on (X, d, m).

Definition 2.4 ([4])**.** (test plan)
We say that $\pi \in \mathcal{P}(C([0, 1]; X))$ is *a test plan* if there exists a constant $c > 0$ such that

$$(e_t)_\# \pi \le cm \quad \text{for every} \quad t \in [0, 1], \qquad \int \mathcal{E}_2[\gamma] d\pi(\gamma) < \infty.$$

Definition 2.5 ([4])**.** (weak upper gradient, Sobolev class and Cheeger energy)

(i) For a Borel function $f : X \to \mathbb{R}$, we say that $G \in L^2(X, m)$ is *a weak upper gradient of* f if the following inequality holds:

$$\int |f(\gamma_1) - f(\gamma_0)| d\pi(\gamma) \le \int \int_0^1 G(\gamma_t) |\dot{\gamma}_t| dt d\pi(\gamma), \tag{2.4}$$

for every test plan π.

(ii) We say that f belongs to the Sobolev class $S^2(X, d, m)$ if there exists $G \in L^2(X, m)$ satisfying (2.4). For $f \in S^2(X, d, m)$, it turns out that there exists a minimal (in the m-a.e. sense) weak upper gradient G and we denote it by $|\nabla f|_w$.

(iii) For $f \in L^2(X, m)$, *the Cheeger energy* $\mathsf{Ch}(f)$ is defined by

$$\mathsf{Ch}(f) = \begin{cases} \dfrac{1}{2} \displaystyle\int_X |\nabla f|_w^2 dm, & \text{if} \quad f \in L^2(X, m) \cap S^2(X, d, m), \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.5}$$

Note that $\mathsf{Ch} : L^2(X, m) \to [0, +\infty]$ is a lower semi-continuous and convex functional but not necessarily quadratic form. Let $W^{1,2}(X, d, m) = L^2(X, m) \cap S^2(X, d, m)$ endowed with the following norm:

$$\|f\|_{W^{1,2}} = \sqrt{\|f\|_{L^2}^2 + 2\mathsf{Ch}(f)}.$$

Note that $(W^{1,2}(X, d, m), \| \cdot \|_{W^{1,2}})$ is a Banach space, but not necessarily a Hilbert space. The Cheeger energy Ch can be defined also as the limit of the integral of local Lipschitz constants. Let $\mathrm{Lip}(X)$ denote the set of real-valued Lipschitz continuous functions on X. For $f \in \mathrm{Lip}(X)$, the local Lipschitz constant $|\nabla f| : X \to \mathbb{R}$ is defined as follows:

$$|\nabla f|(x) = \begin{cases} \limsup_{y \to x} \dfrac{|f(y) - f(x)|}{d(y, x)} & \text{if } x \text{ is not isolated,} \\ 0 \quad \text{otherwise.} \end{cases}$$

Then, for $f \in W^{1,2}(X, d, m)$, we have (see [4])

$$\mathsf{Ch}(f) = \frac{1}{2} \inf \Big\{ \liminf_{n \to \infty} \int_X |\nabla f_n|^2 dm : f_n \in \mathrm{Lip}(X), \int_X |f_n - f|^2 dm \to 0 \Big\}.$$

2.5. RCD(K,∞) **condition.** In this subsection, we recall the definition of metric measure spaces satisfying the RCD(K,∞) conditions following [5]. We also recall several properties satisfied on RCD(K,∞) spaces.

Let (X,d,m) be a metric measure space. Let $\mathcal{P}_2(X,d,m)$ be the subset of $\mathcal{P}_2(X)$ consisting of $\mu\in\mathcal{P}_2(X)$ which is absolutely continuous with respect to m. The relative entropy functional $\mathrm{Ent}_m:\mathcal{P}_2(X,d,m)\to\overline{\mathbb{R}}:=\mathbb{R}\cup\{+\infty\}$ is defined as follows:

$$\mathrm{Ent}_m(\mu)=\int_X\rho\log(\rho)dm,\quad \rho=\frac{d\mu}{dm},$$

where $d\mu/dm$ denotes the Radon–Nikodym derivative. We write $D(\mathrm{Ent}_m):=\{\mu\in\mathcal{P}_2(X,d,m):\mathrm{Ent}_m(\mu)<\infty\}$.

Definition 2.6. (CD(K,∞) and RCD(K,∞))

(i) [Sturm [24], Lott–Villani [17]]
We say that (X,d,m) satisfies *the curvature-dimension condition* CD(K,∞) for $K\in\mathbb{R}$ if, for each $\mu_0,\mu_1\in D(\mathrm{Ent}_m)$, there exists a W_2-geodesic $\{\mu_t\}_{t\in[0,1]}\subset D(\mathrm{Ent}_m)$ connecting μ_0 and μ_1 so that

$$\mathrm{Ent}_m(\mu_t)\le(1-t)\mathrm{Ent}_m(\mu_0)+t\mathrm{Ent}_m(\mu_1)-\frac{K}{2}t(1-t)W_2^2(\mu_0,\mu_1). \tag{2.6}$$

(ii) [Ambrosio–Gigli–Savaré [5, Theorem 5.1] & Ambrosio–Gigli–Mondino–Rajala [3, Theorem 6.1]]
We say that (X,d,m) satisfies *the Riemannian curvature-dimension condition* RCD(K,∞) if the following two conditions hold:
(a) CD(K,∞)
(b) the infinitesimal Hilbertian, that is, the Cheeger energy Ch is linear:

$$2\mathsf{Ch}(u)+2\mathsf{Ch}(v)=\mathsf{Ch}(u+v)+\mathsf{Ch}(u-v). \tag{2.7}$$

for any $u,v\in W^{1,2}(X,d,m)$.

When (X,d,m) satisfies RCD(K,∞) conditions, we define the Dirichlet form (i.e., symmetric closed Markovian bilinear form) $(\mathcal{E},\mathcal{F})$ induced by the Cheeger energy Ch as follows:

$$\mathcal{E}(u,v)=\frac{1}{4}(\mathsf{Ch}(u+v)-\mathsf{Ch}(u-v))\quad u,v\in\mathcal{F}=W^{1,2}(X,d,m). \tag{2.8}$$

By [5, Section 6], the Dirichlet form $(\mathcal{E},\mathcal{F})$ is strongly local, conservative and quasi-regular. Let $\{T_t\}_{t>0}$ be the corresponding heat semigroup associated with $(\mathcal{E},\mathcal{F})$. By general theory of Dirichlet forms (see, e.g., [20]), there exists a unique conservative diffusion process $(\{\mathbb{P}^x\}_{x\in X},\{B_t\}_{t\ge0})$ satisfying

$$\mathbb{E}^x(f(B(t)))=T_tf(x)$$

for all $f\in\mathcal{B}_b(X)\cap L^2(X,m)$, all $t>0$ and quasi-every $x\in X$. By the strong locality of $(\mathcal{E},\mathcal{F})$, there exists a Radon measure $\mu_{\langle\cdot\rangle}$ on X so that (see e.g., [14, §3.2])

$$\mathcal{E}(u,u)=\frac{1}{2}\int_X d\mu_{\langle u\rangle}\quad\forall u\in\mathcal{F}. \tag{2.9}$$

We call $\mu_{\langle\cdot\rangle}$ *the energy measure of* $(\mathcal{E},\mathcal{F})$.

Example 2.7. We give several examples satisfying $\mathrm{RCD}(K,\infty)$ conditions.

(A) **(Ricci limit spaces)** For $N \in \mathbb{N}, K \in \mathbb{R}$ with $N \geq 1$. let $\mathcal{M}(N,K)$ be the set of isomorphism classes of metric measure spaces (M, d_g, m_g) where (M,g) is a complete N-dimensional Riemannian manifold with Ricci $\geq K$, and d_g and m_g denote the Riemannian distance and the Riemannian volume induced by g with $m_g(M) = 1$. It is clear that (M, d_g, m_g) satisfies the infinitesimally Hilbertian condition. By the results of [23, 24], any element in $\mathcal{M}(N,K)$ satisfies $\mathrm{RCD}(K,\infty)$ for any N. Let $\overline{\mathcal{M}(N,K)}$ denote the closure of $\mathcal{M}(N,K)$ with respect to the measured Gromov convergence (see Definition 2.1). Then, by the stability of the $\mathrm{RCD}(K,\infty)$ under the measured Gromov convergence stated below in (i) of Fact 2.8, any element in $\overline{\mathcal{M}(N,K)}$ satisfies $\mathrm{RCD}(K,\infty)$ conditions for any N.

(B) **(Configuration spaces)** Let (M,g) be a complete Riemannian manifold whose Ricci curvature is bounded below by K. Let $\mathcal{M}(M)$ denote the set of Radon measures on M. Let $\Upsilon(M)$ denote the set of locally finite point measures

$$\Upsilon(M) := \{\gamma \in \mathcal{M}(M) : \gamma(K) \in \mathbb{N}_0,\ \forall K \subset M \text{ compact}\}.$$

By [1], there exist a gradient operator ∇^Υ, a divergence operator div^Υ and a Laplace operator Δ^Υ so that, under the Poisson measure π on $\Upsilon(M)$, the gradient ∇^Υ and the divergence div^Υ are dual in $L^2(\Upsilon(M),\pi)$. They constructed a natural energy form on $L^2(\Upsilon(M),\pi)$:

$$\mathcal{E}(F) = \int_\Upsilon |\nabla^\Upsilon F|^2_\Upsilon \pi(d\gamma).$$

This energy form induces the semigroup T_t^Υ and a unique diffusion process on $\Upsilon(M)$. This diffusion process represents the system of independent infinite particles of Brownian motions on the underlying Riemannian manifold M. Let $d_\mathcal{E}$ denote the intrinsic distance associated with $\mathcal{E}$ (possibly taking value $d_\mathcal{E}(\gamma_1,\gamma_2) = \infty$ for some $\gamma_1, \gamma_2 \in \Upsilon(M)$). By [11], the (extended) metric measure space $(\Upsilon(M), d_\mathcal{E}, \pi)$ satisfies $\mathrm{RCD}(K,\infty)$. See [2] for extended metric measure spaces.

(C) **(Hilbert spaces with log-concave measures)** Let H be a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$. A probability measure γ on H satisfies *log-concave condition* if, for all pairs $A, B \subset H$, it holds that

$$\log((1-t)A + tB) \geq (1-t)\log\gamma(A) + t\log\gamma(B).$$

For instance, all measures γ of the following form satisfies the log-concave condition:

$$\gamma = \frac{1}{Z}e^{-V}dx, \quad \text{where } V : H = \mathbb{R}^k \to \mathbb{R} \text{ convex and } Z = \int_{\mathbb{R}^k} e^{-V}dx < +\infty,$$

such as all Gaussian measures and all Gibbs measures on on a finite lattice with a convex Hamiltonian. See [7] for more details. The metric measure space $(H, \|\cdot - \cdot\|, \gamma)$ satisfies $\mathrm{RCD}(0,\infty)$.

We list below several properties of metric measure spaces satisfying the RCD(K,∞) for later use.

Fact 2.8. Assume that (X_n, d_n, m_m) $(n \in \mathbb{N})$ and (X, d, m) satisfy the RCD(K,∞) condition. Then the following statements hold.

(i) (Stability under the mG-convergence)([15, Theorem 7.2])
If $(X_n, d_n, m_n) \overset{mG}{\to} (X_\infty, d_\infty, m_\infty)$, then

$$(X_\infty, d_\infty, m_\infty) \text{ satisfies the RCD}(K,\infty).$$

(ii) ([5, (iv) Theorem 4.18]) Let $\mu_{\langle\cdot\rangle}$ be the energy measure of $(\mathcal{E},\mathcal{F})$ defined in (2.9). Then we have

$$\frac{d\mu_{\langle f\rangle}}{dm} = |\nabla f|_w \quad m\text{-a.e.} \quad \forall f \in \mathcal{F}, \tag{2.10}$$

where $|\nabla f|_w$ means the minimal weak upper gradient defined in (ii) of Definition 2.5.

2.6. **Lyons–Zheng decomposition.** In this subsection, we recall the Lyons–Zheng decomposition theorem for Dirichlet forms. We refer the reader to Lyons–Zheng [18, 19] and Sturm [25, Theorem 5 in Appendix 2].

Let (X, d, m) be a normalized metric measure space. Let $(\mathcal{E},\mathcal{F})$ be a strongly local conservative quasi-regular Dirichlet form on $L^2(X, m)$ and $\mathbb{M} = (\Omega, \mathcal{M}, X_t, P_x)$ be a corresponding diffusion process (see Ma–Röckner [20, Theorem 3.5]). Let us take $\Omega = \{\omega : [0,\infty) \to X : \omega \text{ is continuous}\}$, and, for $\omega \in \Omega$, $X_t(\omega) = \omega(t)$ for $t \geq 0$.

For a path $\omega \in \Omega$, define a time-reversal operator r_t for $t \geq 0$:

$$r_t(\omega)(s) = \begin{cases} \omega(t-s), & \text{if } \ 0 \leq s \leq t, \\ \omega(0), & \text{if } \ s \geq t. \end{cases}$$

We have the following decomposition theorem:

Theorem 2.9 ([18, 19]). *For arbitrary $T > 0$ and for $u \in \mathcal{F}$, we have*

$$u(X_t) - u(X_0) = \frac{1}{2} M_t^{[u]} + \frac{1}{2}(M_{T-t}^{[u]}(r_T) - M_T^{[u]}(r_T)), \quad \mathbb{P}^m\text{-a.e.}, \tag{2.11}$$

for $0 \leq \forall t \leq T$.

Here $M^{[u]}$ is the martingale part of the additive functional $A_t^{[u]} := u(X_t) - u(X_0)$ corresponding to the Fukushima decomposition ([20, §VI Theorem 2.5]).

2.7. **Mosco convergence of Cheeger energies.** In [15], they introduced the L^2-convergence of functions on varying metric measure spaces and showed the Mosco convergence of the Cheeger energies. We recall their results briefly.

Definition 2.10 ([15]). Let (X_n, d_n, m_n) be normalized metric measure spaces. Assume that (X_n, d_n, m_n) converges to $(X_\infty, d_\infty, m_\infty)$ in the mG-sense. Let (X, d) be a complete separable metric space and $\iota_n : \mathrm{supp}[m_n] \to X$ be isometries as in Proposition 2.3. We identify (X_n, d_n, m_n) with $(\iota_n(X_n), d, \iota_{n\#} m_n)$.

(i) We say that $u_n \in L^2(X, m_n)$ *converges weakly to* $u_\infty \in L^2(X, m_\infty)$ if the following hold:

$$\sup_{n\in\mathbb{N}} \int |u_n|^2 \, dm_n < \infty \quad \text{and} \quad \int \phi u_n \, dm_n \to \int \phi u_\infty \, dm_\infty \quad \forall \phi \in C_{bs}(X),$$

where recall that $C_{bs}(X)$ denotes the set of bounded continuous functions with bounded support.

(ii) We say that $u_n \in L^2(X, m_n)$ *converges strongly to* $u_\infty \in L^2(X, m_\infty)$ if u_n converges weakly to u_∞ and the following holds:

$$\limsup_{n\to\infty} \int |u_n|^2 \, dm_n \le \int |u_\infty|^2 \, dm_\infty.$$

Theorem 2.11 (see [15]). *Let (X_n, d_n, m_n) be a sequence of normalized complete separable metric measure spaces satisfying* $\mathrm{CD}(K, \infty)$ *conditions for all $n \in \mathbb{N}$. Assume that $(X_n, d_n, m_n) \overset{mG}{\to} (X_\infty, d_\infty, m_\infty)$ and (X, d) is a complete separable metric space as in Proposition 2.3. Let Ch_n be the Cheeger energy on $L^2(X, m_n)$ for $n \in \overline{\mathbb{N}}$. Then Ch_n* Mosco-converges *to Ch_∞, that is, the following two statements hold:*

(M1) *for every $u_n \in L^2(X, m_n)$ converges weakly to some $u_\infty \in L^2(X, m_\infty)$, the following holds:*

$$\liminf_{n\to\infty} \mathsf{Ch}_n(u_n) \ge \mathsf{Ch}_\infty(u_\infty).$$

(M2) *for every $u_\infty \in L^2(X, m_\infty)$, there exits a sequence $u_n \in L^2(X, m_n)$ such that u_n converges strongly to u_∞ and the following holds:*

$$\limsup_{n\to\infty} \mathsf{Ch}_n(u_n) \le \mathsf{Ch}_\infty(u_\infty).$$

The Mosco convergence of the Cheeger energies implies the convergence of the Heat semigroups. Assume the same conditions as in Theorem 2.11 and Ch_n are linear for any $n \in \mathbb{N}$. Let $\{T_t^n\}_{t>0}$ be the L^2-semigroup corresponding to the Cheeger energy Ch_n.

Theorem 2.12 (see [15]). *Assume the same conditions as in Theorem 2.11 and Ch_n are linear for any $n \in \mathbb{N}$. Then, for any $u_n \in L^2(X, m_n)$ converging strongly to $u_\infty \in L^2(X, m_\infty)$, we have*

$$T_t^n u_n \text{ converges strongly to } T_t^\infty u_\infty \quad (\forall t > 0).$$

3. Proof of Theorem 1.2

The implication (ii) $\implies$ (i) in Theorem 1.2 is trivial by definition of the measured Gromov convergence (see (i) of Remark 1.3)). We only show the implication (i) $\implies$ (ii). The proof consists of two lemmas, the first one shows the tightness (Lemma 3.1) and the second one shows the weak convergence of the laws of finite-dimensional distributions (Lemma 3.2).

Proof of Theorem 1.2.
By Proposition 2.3, there exist a complete separable metric space (X, d) and a family of isometric embeddings $\iota_n : X_n \to X$ such that

$$\iota_{n\#} m_n \to \iota_{\infty\#} m_\infty \quad \text{weakly in } \mathcal{P}(X). \tag{3.1}$$

For $n \in \overline{\mathbb{N}}$, let

$$\mathbb{B}_n := \iota_n(B_\cdot^n)_\# \mathbb{P}_n^{m_n},$$

which is a sequence of probability measures on $\mathcal{P}(C([0,\infty);X))$. Hereafter we identify $\iota_n(X_n)$ with X_n and we omit ι_n. To show $\mathbb{B}_n \to \mathbb{B}_\infty$ weakly, it is enough to show the tightness and the weak convergence of the finite-dimensional distributions. Note that $\mathrm{Lip}_b(X)$ is dense in $C_b(X)$ with respect to the uniform topology on compact sets. Thus, by [12, Theorem 3.9.1], it suffices for the tightness of $\{\mathbb{B}_n\}_n$ to show that, for any $h \in \mathrm{Lip}_b(X)$,

$$h(B^n_\cdot)_\# \mathbb{P}^{m_n}_n \quad \text{is tight in} \quad \mathcal{P}(C([0,\infty);\mathbb{R})). \tag{3.2}$$

We note that, although [12, Theorem 3.9.1] gives sufficient conditions for tightness only in the càdlàg space $D([0,\infty);X)$, since each Brownian motion $\mathbb{B}_n$ supports on the space of continuous paths $C([0,\infty);X)$, the weak convergence (3.2) implies the tightness of $\{\mathbb{B}_n\}_{n\in\mathbb{N}}$ also on $C([0,\infty);X)$. See, e.g., [13, Lemma 5 in Appendix] for this point. We also note that, in [12, Theorem 3.9.1], we have to check *the compact containment condition*, and it can be checked by the weak convergence of the laws of finite-dimensional distributions of $\{\mathbb{B}_n\}_{n\in\mathbb{N}}$. See the proof of [12, Corollary 3.9.2]. Thus what we should prove are (3.2) for any $h \in \mathrm{Lip}_b(X)$ and the weak convergence of the laws of finite-dimensional distributions of $\{\mathbb{B}_n\}_{n\in\mathbb{N}}$.

Let $\mathbb{B}^h_n := h(B^n_\cdot)_\#\mathbb{P}^{m_n}_n$ for $h \in \mathrm{Lip}_b(X)$. By the above argument, it suffices for the desired result to show

(A) $\{\mathbb{B}^h_n\}_{n\in\mathbb{N}}$ is tight in $\mathcal{P}(C([0,\infty),\mathbb{R}))$ with respect to the weak topology;

(B) Convergence of the finite-dimensional distributions: For any $k \in \mathbb{N}$, $0 = t_0 < t_1 < t_2 < \cdots < t_k < \infty$ and $g_1, g_2, ..., g_k \in C_b(X)$, the following holds:

$$\mathbb{E}^{m_n}[g_1(B^n_{t_1}) \cdots g_k(B^n_{t_k})] \overset{n\to\infty}{\to} \mathbb{E}^{m_\infty}[g_1(B^\infty_{t_1}) \cdots g_k(B^\infty_{t_\infty})]. \tag{3.3}$$

We first show (A), that is, the following statement holds:

Lemma 3.1. *$\{\mathbb{B}^h_n\}_{n\in\mathbb{N}}$ is tight in $\mathcal{P}(C([0,\infty),\mathbb{R}))$.*

Proof. Since m_n converges weakly to m_∞ in (X,d), the laws of the initial distributions $\{h(B^n_0)_\#\mathbb{P}^{m_n}_n\}_{n\in\mathbb{N}} = \{h_\# m_n\}_{n\in\mathbb{N}}$ is clearly tight in $\mathcal{P}(\mathbb{R})$. For $\delta > 0$, let us define

$$L^{n,h}_{\eta,T}(x) := \mathbb{P}^x_n\Big(\sup_{\substack{0\le s,t\le T\\ |t-s|\le\eta}} |h(B_t) - h(B_s)| > \delta\Big).$$

Thus our objective is to show

$$\lim_{\eta\to 0} \sup_{n\in\mathbb{N}} m_n(L^{n,h}_{\eta,T}) = 0, \tag{3.4}$$

for any $T > 0$. Here recall that we mean $m_n(f) := \int_{X_n} f dm_n$ for $f \in L^1(X_n, m_n)$. By Theorem 2.9, we have

(3.5)
$$h(B^n_t) - h(B^n_s) = \frac{1}{2}(M^{[h]}_t - M^{[h]}_s) + \frac{1}{2}(M^{[h]}_{T-t}(r_T) - M^{[h]}_{T-s}(r_T)), \quad \mathbb{P}^{m_n}\text{-a.e.},$$

for $0 \le t \le T$.

Then by time-symmetry (see [14, Lemma 5.7.1]), we have

(3.6)

$$
\begin{aligned}
&\mathbb{P}^{m_n}\big(\sup_{\substack{0\le s,t\le T\\|t-s|\le\eta}} |h(B^n_t)-h(B^n_s)|>\delta\big)\\
&\le \mathbb{P}^{m_n}\big(\sup_{\substack{0\le s,t\le T\\|t-s|\le\eta}} \big|M^{[h],n}_t-M^{[h],n}_s\big|>\delta\big)+\mathbb{P}^{m_n}\big(\sup_{\substack{0\le s,t\le T\\|t-s|\le\eta}} \big|M^{[h],n}_{T-t}(r_T)-M^{[h],n}_{T-s}(r_T)\big|>\delta\big)\\
&=2\mathbb{P}^{m_n}\big(\sup_{\substack{0\le s,t\le T\\|t-s|\le\eta}} \big|M^{[h],n}_t-M^{[h],n}_s\big|>\delta\big).
\end{aligned}
$$

Since $M^{[h],n}$ is a continuous martingale, by the martingale representation theorem, there exists the one-dimensional Brownian motion $\mathbf{B}^n(t)$ on an extended probability space $(\widetilde{\Omega},\widetilde{\mathcal{M}},\widetilde{\mathbb{P}}^x_n)$ where $M^{[h],n}$ is represented as a time-changed Brownian motion with respect to the quadratic variation $\widetilde{\mathbb{P}}^x_n$-a.s, q.e. $x\in X_n$ (see, e.g., Ikeda–Watanabe [16, Chapter II Theorem 7.3']). That is, for q.e. $x\in X_n$,

(3.7)

$$
M^{[h],n}_t=\mathbf{B}^n(\langle M^{[h],n}\rangle_t)=\mathbf{B}^n\Big(\int_0^t \frac{d\mu^n_{\langle h\rangle}}{dm_n}(B^n_u)du\Big)=\mathbf{B}^n\Big(\int_0^t |\nabla h|^2_w(B^n_u)du\Big)\quad \widetilde{\mathbb{P}}^x_n\text{-a.s.}
$$

In the last equality above, we used (ii) of Fact 2.8. Since $|\nabla h|_w\le \mathrm{Lip}(h)$, we have

$$
\begin{aligned}
&\{\omega\in\widetilde{\Omega}:\sup_{\substack{0\le s,t\le T\\|t-s|\le\eta}}\big|M^{[h],n}_t-M^{[h],n}_s\big|>\delta\}\\
&=\{\omega\in\widetilde{\Omega}:\sup_{\substack{0\le s,t\le T\\|t-s|\le\eta}}\Big|\mathbf{B}^n\Big(\int_0^t|\nabla h|^2_w(B^n_u)du\Big)-\mathbf{B}^n\Big(\int_0^s|\nabla h|^2_w(B^n_u)du\Big)\Big|>\delta\}\\
&\subset\{\omega\in\widetilde{\Omega}:\sup_{\substack{0\le s,t\le \mathrm{Lip}(h)^2T\\|t-s|\le\mathrm{Lip}(h)^2\eta}}\big|\mathbf{B}^n(t)-\mathbf{B}^n(s)\big|>\delta\}.
\end{aligned}
$$

Let $\mathbb{W}$ be the standard Wiener measure on $C([0,\infty);\mathbb{R})$. Let

$$
\theta(\eta,h):=\mathbb{W}_n\big(\sup_{\substack{0\le s,t\le \mathrm{Lip}(h)^2T\\|t-s|\le\mathrm{Lip}(h)^2\eta}}|\omega(t)-\omega(s)|>\delta\big).
$$

Noting $m_n(X_n)=1$, by (3.6), we have, for any $T>0$,

$$
\sup_{n\in\mathbb{N}} m_n(L^{n,h}_{\eta,T})\le\sup_{n\in\mathbb{N}}2\mathbb{P}^{m_n}\big(\sup_{\substack{0\le s,t\le T\\|t-s|\le\eta}}\big|M^{[h]}_t-M^{[h]}_s\big|>\delta\big)\le 2\theta(\eta,h)\sup_{n\in\mathbb{N}}m_n(X_n)\overset{\eta\to0}{\to}0.
$$

Thus we have the desired result (3.4).

□

Now we show (B), that is, the following statement holds:

Lemma 3.2. *For any $k\in\mathbb{N}$, $0=t_0<t_1<t_2<\cdots<t_k<\infty$ and $g_1,g_2,...,g_k\in C_b(X)$, the following holds:*

$$
\mathbb{E}^{m_n}[g_1(B^n_{t_1})\cdots g_k(B^n_{t_k})]\overset{n\to\infty}{\to}\mathbb{E}^{m_\infty}[g_1(B^\infty_{t_1})\cdots g_k(B^\infty_{t_k})]. \tag{3.8}
$$

Proof. Recall that we have

$$T_t^n f(x) = \mathbb{E}_n^x[f(B_n(t))],$$

for q.e. $x \in X_n$, and $f \in L^2(X_n, m_n) \cap \mathcal{B}_b(X_n)$. For $g \in C_b(X)$, we set

$$g^{(n)} := g|_{\iota_n(X_n)} := \begin{cases} g & \text{on} \quad \iota_n(X_n), \\ 0 & \text{otherwise.} \end{cases}$$

By $m_n(X_n) = 1$, we have $g^{(n)} \in L^2(X, m_n)$ for all $g \in C_b(X)$.

By using the Markov property, for all $n \in \overline{\mathbb{N}}$, we have

$$\begin{aligned}
&\mathbb{E}_n^x[g_1(B_{t_1}^n) \cdots g_k(B_{t_k}^n)] \\
&= T_{t_1-t_0}^n\Big(g_1^{(n)} T_{t_2-t_1}^n\Big(g_2^{(n)} \cdots T_{t_k-t_{k-1}}^n g_k^{(n)}\Big)\Big)(x) \\
&=: \mathcal{T}_k^n(x),
\end{aligned}$$

for q.e. $x \in X_n$. By [5, Theorem 6.1 (iii)], we have $\mathcal{T}_k^n \in C_b(X_n)$ and

$$\sup_{n\in\mathbb{N}} \|\mathcal{T}_k^n\|_\infty \le \sup_{n\in\mathbb{N}} \prod_{i=1}^k \|g_i^{(n)}\|_\infty \le \prod_{i=1}^k \|g_i\|_\infty < \infty, \tag{3.9}$$

which is bounded uniformly in n. By Theorem 2.12 and the same argument in [26, Lemma 3.3 & (a) in Lemma 3.5]), we have

$$\mathcal{T}_k^n \to \mathcal{T}_k^\infty \text{ strongly in the sense of Definition 2.10.} \tag{3.10}$$

Since m_n converges weakly to m_∞ in $\mathcal{P}(X)$, for any $\varepsilon > 0$, there exists a compact set $K \subset X$ so that

$$\sup_{n\in\mathbb{N}} m_n(K^c) < \varepsilon. \tag{3.11}$$

Thus, for any $\delta > 0$, there exists a compact set $K \subset X$ so that, noting (3.9) and (3.11), we have

$$\sup_{n\in\mathbb{N}}\Big|\int_X \mathcal{T}_k^n \, dm_n - \int_K \mathcal{T}_k^n \, dm_n\Big| \le \Big(\prod_{i=1}^k \|g_i\|_\infty\Big) \sup_{n\in\mathbb{N}} m_n(K^c) < \delta, \tag{3.12}$$

for any $n \in \overline{\mathbb{N}}$. Take $r > 0$ so that $K \subset B_r(x_0) := \{x \in X : d(x_0, x) < r\}$ for some fixed $x_0 \in X$. Let $\widetilde{\mathbf{1}}_r^R$ denote the following truncated function: $(r < R)$

$$\widetilde{\mathbf{1}}_r^R(x) = \begin{cases} 1 & x \in B_r(x_0), \\ 1 - \dfrac{d(x, B_r(x_0))}{R - r} & x \in B_R(x_0) \setminus B_r(x_0), \\ 0 & \text{o.w.} \end{cases}$$

Then $\widetilde{\mathbf{1}}_r^R \in C_{bs}(X)$. Thus, by (3.10), Theorem 2.12 and (3.12), for any $\delta > 0$, we can take so large $r > 0$ that

$$
\begin{aligned}
&\left|\mathbb{E}^{m_n}[g_1(B^n_{t_1})\cdots g_k(B^n_{t_k})] - \mathbb{E}^{m_\infty}[g_1(B^\infty_{t_1})\cdots g_k(B^\infty_{t_k})]\right| \\
&= \Big|\int_X \mathcal{T}^n_k \, dm_n - \int_X \mathcal{T}^\infty_k \, dm_\infty\Big| \\
&= \Big|\int_X \mathcal{T}^n_k \, dm_n - \int_X \widetilde{\mathbf{1}}_r^R \mathcal{T}^n_k \, dm_n + \int_X \widetilde{\mathbf{1}}_r^R \mathcal{T}^n_k \, dm_n - \int_X \widetilde{\mathbf{1}}_r^R \mathcal{T}^\infty_k \, dm_\infty \\
&\quad + \int_X \widetilde{\mathbf{1}}_r^R \mathcal{T}^\infty_k \, dm_\infty - \int_X \mathcal{T}^\infty_k \, dm_\infty\Big| \\
&\le \delta + \Big|\int_X \widetilde{\mathbf{1}}_r^R \mathcal{T}^n_k \, dm_n - \int_X \widetilde{\mathbf{1}}_r^R \mathcal{T}^\infty_k \, dm_\infty\Big| + \delta \\
&\overset{n\to\infty}{\to} 2\delta.
\end{aligned}
$$

Here in the fifth line above, the first δ of the first term follows from (3.12), and the second δ of the third term follows just from the tightness of the single measure m_∞. Thus we have completed the proof. □

Now we resume the proof of Theorem 1.2.

Proof of Theorem 1.2:
By Lemma 3.1 and Lemma 3.2, we have completed the proof of Theorem 1.2. □

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